4-DAY WORKSHOP: JUNE 19/22 2019

"NONLINEAR AVERAGING AND PDES "

ON THE OCCASION OF THE 60TH BIRTHDAY OF J.MANFREDI

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Description of the workshop. The workshop, organized by Julio Rossi (U.Buenos Aires) and Bianca Stroffolini(U.Naples) aimed at presenting recent and promising achievements in this wide area. There were 30 different talks (all of 40 minutes) with a broad geographic representation (ranging from Japan, Norway, Finland, USA, Argentina, Germany, France, Italy, etc) and a variety of research fields (like variational methods, regularity, probability, etc) attended, each revealing different methodology, interests, and level of abstraction. Among the many themes presented during the workshop, we record here:

- p-Laplacian in subelliptic structures or even metric measure spaces;
- doubly nonlinear parabolic equations;
- fractional p-Laplacian;
- games for the p-Laplacian;
- variational problems: Convexity and uniqueness;
- eigenvalues problems;
- optimal regularity for the ∞-Laplacian free boundary problem;
- optimal transport;
- nonlinear Calderon-Zygmund theory.

The resulting atmosphere was very productive given rise to new and promising lines of research, establishing potential collaborations between the assistants. For example, let us mention as part of the main open problems in this field:

- optimal local $C^{1,\alpha}$ regularity for the p-Laplacian;
- regularity estimates up to the boundary;
- The validity of the strong comparison principle for the p-Laplacian;
- The validity of the unique continuation property for the p-Laplacian;
- the study of the fractional p-Laplacian in sub elliptic geometries;
- games for other equations related to the p-Laplacian;
- optimal mass transport problems in different geometries;
- free boundary problems.

Mathematical background. The main goal of this workshop was to gather researchers interested in the the study of the $p-$Laplacian (taking advantage of the opportunity to celebrate the 60 anniversary of one of the best specialists in the subject, Juan Manfredi). The p-Laplacian is a well-known partial differential equation that attracted interest in the mathematical community both by its applications and also from its mathematical challenges. The formula for the operator reads as

$$-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

and it is naturally associated with critical points of the $p-$energy

$$E(u) = \frac{1}{p}\int |\nabla u|^p.$$

The p-Dirichlet integral can be pushed to a more general class of convex functionals with a great lack of uniform convexity, the so called orthotropic ones,that can be seen as limit of congested traffic problems on very dense networks, Brasco. Another generalization is provided by strongly quasiconvex variational integrals. It is well-known that their minimizers need not be regular nor unique. However, if a suitable Garding type inequality is assumed for the variational integral, then both regularity and uniqueness of minimizers can be restored under natural smallness conditions on the data, Kristensen.

Zhikov was one of the pioneer in studying Lavrentiev phenomenon for double phase functionals. He used a checkerboard setup to construct an example, where the exponent crosses the dimension. New examples on the Lavrentiev gap phenomenon using fractals have been presented by Diening. He applied this technique to the setting of minimizer with variable exponents. He also showed that the Lavrentiev gap occurs for any range of exponents. This gives a negative answer to the conjecture that the dimension plays a critical threshold for the exponent.

In the last decade, models for nematic elastomers and magnetoelasticity has been extensively studied. These models consider both an elastic term where a polyconvex energy density is composed with an unknown state variable defined in the deformed configuration, and a functional corresponding to the nematic energy (or the exchange and magnetostatic energies in magnetoelasticity) where the energy density is integrated over the deformed configuration. In order to obtain the desired compactness and lower semicontinuity, one has to face the regularity requirement that maps create no new surface. Henao and Mora Corral have investigated the notions of surface measure, topological image and geometric image, degree in order to define a class of sense preserving local homeomorphism in Sobolev spaces. However, so far, the optimal result seems to hold for maps whose gradients are in an Orlicz class with an integrability just above the space dimension minus one. Fine properties of orientation-preserving maps satisfying that regularity requirement (namely, being weakly 1-pseudomonotone,H^{n-1} continuous, a.e. differentiable, and a.e. locally invertible) are still valid in the Orlicz-Sobolev setting.

The p-Laplacian operator can be seen as a prototype of a nonlinear, degenerate or singular diffusion operator that can be understood as divergence and non-divergence type equation. It shares both the variational structure, as the usual linear Laplacian and the nonvariational one, as for the fully nonlinear equations. The two theories of weak solutions (well suited for divergence form equations) and viscosity solutions (better suited for the understanding of fully nonlinear problems) have been identified for the p-laplacian in a seminal paper by Juutinen, Linqvist , Manfredi back in the 90's.

This equation also makes sense in a more general framework considering different geometric structures given rise to a very rich and difficult area called sub elliptic problems. In this context, there have been presented results about the H-convex envelope (horizontal) in the Heisenberg Group with applications to semilinear equations (Liu and Zhao), geometrical properties as starshapeness inherited by level sets of solutions of PDEs, Dragoni. In a more general framework, as PI metric spaces, the existence of the Green function for the Q-Laplacian has been presented by Capogna.

The p-Laplacian operator can be studied also from the point of view of Nonlinear Potential Theory. Iwaniec investigated in the paper Projections onto gradient fields and L^p-estimates for degenerated elliptic operators, Studia Math. 75 (1983),the higher integrability of the solution of the p-Laplacian system versus the higher integrability of the right-hand side in divergence form. Lately, Di Benedetto and Manfredi were using a different technique to get similar estimates and also BMO estimates for p-Laplacian systems. Calderon-Zygmund estimates have been presented by Cianchi in Campanato, Lorentz and Orlicz spaces for solutions of the p-Laplacian system with right-hand side in divergence form. They are based on pointwise estimates concerning sharp maximal operators. Here it appears as natural quantity to bound the density $|Du|^{p-2}Du$. From another point of view, Mingione considered potential estimates for vectorial solutions of p-Laplacian systems with right-hand side a Borel measure with finite total mass. Here the Riesz and Wolff potentials come into play for the solution and the gradient respectively. The estimates are sharp in the sense that they reproduce the exact analogues when restricted to the linear potential theory.

Consider the p-Poisson equation :

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = f \in L^\infty(B_1)$$

The $C^{p'}$- conjecture claims that any weak solution lies in $C^{p'}(B_{\frac{1}{2}})$, where p' is the conjugate of p. The result holds true in the plane. In higher dimension there is a partial answer: radially symmetric solutions or solutions with no saddle critical points, Urbano.

Recently, there has been a probabilistic (game-theoretic) approach towards p-Laplacian equations. The drawback has been the search of asymptotic mean value formulas, and possibly new proofs of classical results like Harnack's inequality.

The analysis of the connection between game theory and the $p-$Laplacian was motivated by the study of different variants of a special kind of games called Tug-of-War games in the literature. These games have lead to a new chapter in the rich history of results connecting differential equations and probability theory.

The fundamental works by Doob, Feller, Hunt, Kakutani, Kolmogorov and many others show the deep connection between classical potential theory and probability theory. The main idea that is behind this relation is that harmonic functions and martingales have something in common: the mean value formulas. This relation is also quite fruitful in the non-linear case and the Tug-of-War games are a clear evidence of this fact.

At the end of the decade of the 80s the mathematician David Ross Richman proposed a new kind of game that lies in between the classical games introduced by Von Neumann and Morgenstern, and the combinatoric games studied by Zermelo, Lasker and Conway among others. Here two players are in contest in an arbitrary combinatoric game (Tic-tac-toe, chess, checkers, etc.), but each one of them has a certain amount of money that modifies the rules of the game: at each turn the players bid and the one who offered more win the right to make the next move. In case that both players bid the same amount the turn can be decided by a coin toss.

At the end of the 90s this kind of games, known as Richman's games were studied. They have been translated into a diffusion problem on a graph: the nodes stand for the positions of the game and the links between the nodes are the allowed moves, a token is moved by one of the players according to who wins the bidding. The game ends when the token arrives to the nodes of the graph labelled as terminal ones and there a certain boundary datum says how much the first player gets (that is the amount of money that the other player pays).

Among the several variants of these games, an interesting case is when the turn is decided at random tossing a fair coin at each turn, getting rid in this way of the bidding mechanism. This idea gives rise to the game called Tug-of-War introduced by Peres, Schramm, Sheffield and Wilson. In that reference this game was studied and a novel connection with PDEs was found. More concretely, with the $\infty-$Laplacian, an operator that appears naturally in a completely different context since it is associated to the minimal Lipschitz extension problem.

Being one of the triggers of this workshop, we briefly describe now the Tug-of-War game game introduced by Peres, Schram, Sheffield and

Wilson. The Tug-of-War game is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one of them are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome).

Consider a bounded domain $\Omega \subset \mathbb{R}^N$ and a fixed $\epsilon > 0$. At an initial time, a token is placed at a point $x_0 \in \Omega$. Players I and II play as follows: They toss a fair coin (with the same probability for heads and tails) and the winner of the toss moves the game token to any point x_1 of his choice at distance less than ε of the previous position, that is he chooses $x_1 \in B_\varepsilon(x_0)$. Then, they continue playing from x_1 with the same rules, at each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in B_\epsilon(x_{k-1})$.

This procedure yields a sequence of game states $x_0, x_1, \ldots$. Once the game position leaves Ω, let say at the τ-th step, the game ends. At that time the token will be at the final position x_τ in $\mathbb{R}^N \setminus \Omega$. A *final payoff function* $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ is given. At the end of the game Player II pays to Player I the amount given by $g(x_\tau)$, that is, Player I have earned $g(x_\tau)$ while Player II have earned $-g(x_\tau)$.

A strategy S_I for Player I is a choice of the next position $x_{k+1} \in B_\varepsilon(x_k)$ of the game at every ocation provided he/she wins the coin toss. When the two players fix their strategies the movements depend only on the coin tosses and hence we can compute its expected value. Now, for each $x_0 \in \Omega$ we can consider the expected payoff $u^\epsilon(x_0)$ for the game starting at x_0 assuming that both players play optimal. To do this we just compute the infimum among all possible strategies for Player II and the supremum among strategies for Player I of the expected value (here we need to mention that we have to penalize the use of strategies that produce games that never end with positive probability). This is what we call the *game value* and we denote it by u^ϵ. Hence, for each ϵ, we have a function $u^\epsilon : \overline{\Omega} \to \mathbb{R}$ that depends only on the initial position of the game. It can be proved that there exists a continuous function $u : \bar{\Omega} \to \mathbb{R}$ such that $u^\varepsilon \to u$ as $\varepsilon \to 0$, and that u satisfies the PDE

$$-\Delta_\infty^H u = -|\nabla u|^{-2} \langle D^2 u \nabla u; \nabla u \rangle = 0 \qquad \text{in } \Omega,$$

with the boundary condition

$$u = g$$

on $\partial\Omega$. The key argument to show this convergence result is to prove that the value of the game u^ε verifies an equation inside Ω that is called the Dynamic Programming Principle (DPP) in the literature. In this case, for the Tug-of-War game, the DPP reads as

$$u^\varepsilon(x) = \frac{1}{2} \sup_{B_\varepsilon(x)} u^\varepsilon + \frac{1}{2} \inf_{B_\varepsilon(x)} u^\varepsilon$$

for $x \in \Omega$, with $u^\varepsilon(x) = g(x)$ for $x \notin \Omega$. Then, we prove that there exists a continuous function u that is a uniform limit of a subsequence of these game values u^ε as $\varepsilon \to 0$, that is, we show that

$$u^\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega}.$$

The key ingredient to prove this uniform convergence is an Arzela-Ascoli type lemma in which one shows that the family u^ε is uniformly bounded and asymptotically equicontinuous (here we remark that the value functions are not continuous in general).

Then, to obtain the limit equation, we just have to pass to the limit (in the viscosity sense) in the DPP to obtain that any uniform limit of the value functions u^ε is a viscosity solution to the limit PDE. Finally, the uniqueness of solution to the PDE will allow us to conclude the convergence of the whole sequence.

When we deal with partial differential equations we have to mention the concept of solution that we are considering. The theory for second order operators in divergence form is naturally associated to the concept of weak solutions in Sobolev spaces; however, when one deals with fully nonlinear equations that are not in divergence form, the use of viscosity solutions seems more appropriate. This notion of solution was introduced by Crandall and Lions in the 80s.

After this seminal work many versions of the game were considered and many results obtained. It turns out that this strategy of finding a game, showing that its value functions verifies a DPP and then pass to the limit, is quite flexible and allows to deal with several different PDEs elliptic or parabolic including free boundary problems.