Regularity for solutions of restricted mean value properties

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In 1979 Krylov and Safonov proved the Harnack inequality for solutions of the elliptic PDE in non-divergence form $\mathrm{Tr}\{A(x)\cdot D^2u(x)\}=0$, where $A(\cdot)$ is a symmetric and uniformly elliptic matrix-valued function with measurable coefficients in $\Omega\subset\mathbb{R}^n$. Then the interior Hölder regularity of solutions followed as an immediate consequence. On the other hand, it turns out that viscosity solutions to this equation can be asymptotically characterized by means of a restricted mean value property over a suitable family of ellipsoids. This establishes a connection between the PDE and a stochastic process describing a random walk in which the next step is chosen inside an (space-dependent) ellipsoid. Then the value functions of this process satisfy a *dynamic programming principle (DPP)*: a functional equation which reads as

$$u_\varepsilon(x)=\fint_{E_\varepsilon(x)} u_\varepsilon(y)\ dy,$$

where $E_\varepsilon(x)$ stands for an ellipsoid centered at x, with size controlled by $\varepsilon>0$ and shape determined by $A(x)$.

In a joint work with M. Parviainen, we show that, assuming some uniform bound for the distortion of the ellipsoids, the solutions of the DPP are (locally) asymptotically Hölder continuous, that is

$$|u_\varepsilon(x)-u_\varepsilon(y)|\leq C\big(|x-y|^\alpha+\varepsilon^\alpha\big)$$

for all $x,y\in B_r\subset\Omega$, where $C>0$ and $0<\alpha<1$ do not depend on ε. Moreover, by letting $\varepsilon\to 0$, this provides a Hölder estimate for the limit function, which turns out to be a viscosity solutions to $\mathrm{Tr}\{A(x)\cdot D^2u(x)\}=0$.

In addition, in this talk we also review some other regularity results for solutions of restricted mean value properties related to partial differential operators such as the p-Laplacian and the normalized $p(x)$-Laplacian.