A SENSE PRESERVING SOBOLEV HOMEOMORPHISM WITH NEGATIVE JACOBIAN ALMOST EVERYWHERE.

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In 2001 Hajłasz asked whether a Jacobian of a Sobolev homeomorphism in $W^{1,p}_{\text{loc}}$, $1 \leq p < n-1$ can change sign. That is, whether there is a homeomorphims such that $J_f > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure. In [6] the authors answered this question for a range of integrability exponent. More precisely

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n \leq 3$, be a domain and let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ be a sense preserving homeomorphism. Then $J_j \geq 0$ a.e.*

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a domain and let $f : \Omega \to \mathbb{R}^n$ be a sense preserving homeomorphism such that $Df \in L^{[n/2],1}$. Then $J_j \geq 0$ a.e.*

where $[x]$ stands for the greatest integer less or equal to x. We recall that $L^{[n/2]} \subset L^{[n/2],1}$ so that Theorem 2 answers the question of Hajłasz for $p > [n/2]$.

In [3] the authors gave a positive answer under some additional assumptions. Indeed they provided an example of a homeomorphism f in $W^{1,1}((-1,1)^n, \mathbb{R}^n)$ such that $J_f > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure. Moreover f maps null set into null set, that is, f satisfies the Lusin condition. This example is generalized in [1]:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 4$ and $1 \leq p < [n/2]$, then there is a homeomorphism $f \in W^{1,p}((-1,1)^n, \mathbb{R}^n)$ such that $J_f > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure. Moreover f satisfies the Lusin condtion.*

The examples constructed in [7] and [1] gave new ideas to answer another question posed by Hajłasz (see [5]), that is:

It is possible to constructed a homeomorphism $\varphi :\to [0,1]^n$ which is approximately diffentiable a.e., has the Lusin property, euquals to the identity on the boundary (and hence it is sense preserving in the topological sense) but $J_\varphi < 0$ a.e. ?

In [2] the authors focus to the previous question without assuming Sobolev regularity of the homeomorphism.

Following the main idea of [1] we would like to ansewer to the question of Hajłasz constructing some homeomorphism in a Sobolev class. Our main result is the following:

Theorem 4. *There exists a Sobolev homeomorphism* $f \in W^{1,p}((-1,1)^4,(-1,1)^4)$ *such that* $f(x) = x$ *for every* $x \in \partial(-1,1)^4$ *but* $J_f(x) < 0$ *for a.e.* $x \in (-1,1)^4$.

This result is deeply connected with the problem of the approximation of Sobolev homeomorphism, indeed:

Corollary 5. *Set* $\tilde{f}(x_1,x_2,x_3,x_4) = f(-x_1,x_2,x_3,x_4)$ *where* f *is from Theorem 4. Then* $J_{\tilde{f}}(x) > 0$ *a.e. but there are no diffeomorphisms (or piecewise affine homeomorphisms)* f_k *such that* $f_k \to \tilde{f}$ *in* $W^{1,p}$.

The previous results are in collaboration with D. Campbell and S. Hencl.

References

[1] Campbell D., Hencl S. and TengvalL V. *Approximation of* $W^{1,p}$ *Sobolev homeomorphism by diffeomorphisms and the signs of the Jacobian* Adv. Math. vol 331, 748–829, (2018).

[2] Goldstein P. and Hajlasz P. *A measure and orientation preserving homeomorphism of a cube with Jacobian equal* -1 *almost everywhere* Arch. Ration. Mech. Anal. vol 225, 65–88, (2017).

[3] Goldstein P. and Hajlasz P. *Jacobians of* $W^{1,p}$ *homeomorphism, case* $p = [n/2]$ preprint 2018.

[4] Goldstein P. and Hajlasz P. *Modulus of continuity of orientation preserving approximately differentiable homeomorphisms with a.e. negative Jacobian* preprint 2018

[5] Hencl S. and Koskela P. **Lectures on mappings of finite distortion** Lecture Notes in Mathematics 2096, (2014).

[6] Hencl S. and Malý J. *Jacobians of Sobolev homeomorphisms* Calc. Var., vol 38, 233–242, (2010).

[7] Hencl S. and Vejnar B. *Sobolev homeomorphisms that cannot be approximated by diffeomorphisms in* $W^{1,1}$ Arch. Rational Mech. Anal., vol 219, 183–202, (2016).

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