A SENSE PRESERVING SOBOLEV HOMEOMORPHISM WITH NEGATIVE JACOBIAN ALMOST EVERYWHERE.

LUIGI D’ONOFRIO

In 2001 Hajlasz asked whether a Jacobian of a Sobolev homeomorphism in $W^{1,p}_{\text{loc}}$, $1 \leq p < n - 1$ can change sign. That is, whether there is a homeomorphism such that $J_f > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure. In [6] the authors answered this question for a range of integrability exponent. More precisely

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$, $n \leq 3$, be a domain and let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ be a sense preserving homeomorphism. Then $J_f \geq 0$ a.e.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a domain and let $f : \Omega \to \mathbb{R}^n$ be a sense preserving homeomorphism such that $Df \in L^{[n/2],1}$. Then $J_f \geq 0$ a.e.

where $[x]$ stands for the greatest integer less or equal to $x$. We recall that $L^{[n/2]} \subset L^{[n/2],1}$ so that Theorem 2 answers the question of Hajlasz for $p > [n/2]$.

In [3] the authors gave a positive answer under some additional assumptions. Indeed they provided an example of a homeomorphism $f$ in $W^{1,1}((-1,1)^n, \mathbb{R}^n)$ such that $J_f > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure. Moreover $f$ maps null set into null set, that is, $f$ satisfies the Lusin condition. This example is generalized in [1]:

**Theorem 3.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 4$ and $1 \leq p < [n/2]$, then there is a homeomorphism $f \in W^{1,p}((-1,1)^n, \mathbb{R}^n)$ such that $J_f > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure. Moreover $f$ satisfies the Lusin condition.

The examples constructed in [7] and [1] gave new ideas to answer another question posed by Hajlasz (see [5]), that is:

Is it possible to constructed a homeomorphism $\varphi : \to [0,1]^n$ which is approximately differentiable a.e., has the Lusin property, equals to the identity on the boundary (and hence it is sense preserving in the topological sense) but $J_\varphi < 0$ a.e.?
In [2] the authors focus to the previous question without assuming Sobolev regularity of the homeomorphism. Following the main idea of [1] we would like to answer to the question of Hajlasz constructing some homeomorphism in a Sobolev class. Our main result is the following:

**Theorem 4.** There exists a Sobolev homeomorphism $f \in W^{1,p}((-1,1)^4, (-1,1)^4)$ such that $f(x) = x$ for every $x \in \partial(-1,1)^4$ but $J_f(x) < 0$ for a.e. $x \in (-1,1)^4$.

This result is deeply connected with the problem of the approximation of Sobolev homeomorphism, indeed:

**Corollary 5.** Set $\tilde{f}(x_1, x_2, x_3, x_4) = f(-x_1, x_2, x_3, x_4)$ where $f$ is from Theorem 4. Then $J_{\tilde{f}}(x) > 0$ a.e. but there are no diffeomorphisms (or piecewise affine homeomorphisms) $f_k$ such that $f_k \to \tilde{f}$ in $W^{1,p}$.

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**References**


